Optimal In-Plane Orbital Evasive Maneuvers Using Continuous Low Thrust Propulsion

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Introduction

IN recent years, much research has been accomplished on optimal maneuvering in space using impulsive and continuous thrust to address problems such as orbit transfer, stationkeeping, and rendezvous. Very little research, however, has addressed the optimal intercept/avoidance problem, which relates directly to antisatellite, space reconnaissance, and collision-avoidance objectives. The optimal intercept/avoidance problem is becoming more relevant to spacecraft mission planning because of current and emerging capabilities to intercept and avoid interception. Where time permits, any target under intercept might be maneuvered to attempt to prevent a successful intercept. This option motivates this study of optimal evasive-maneuver strategies using a continuous low thrust propulsion system.

Problem Statement

The problem considered here is that of a target satellite in geosynchronous orbit that must be maneuvered to avoid intercept by a coasting object on a collision course. After the evasive maneuver is completed, the target satellite is to return to its nominal orbit in minimum time. Both maneuvers are to be done with a continuous low thrust propulsion system that might normally be used for stationkeeping or long-term orbit changes. In addition, each maneuver is restricted to remain in the plane of the nominal orbit.

The position and velocity of the target satellite are measured in a geocentric equatorial XY coordinate system. The translational motion of the satellite is assumed to occur in an inverse square gravitational field and during maneuvers under the additional action of a constant low thrust constant mass flow rate propulsion system. The thrust vector direction in the XY plane is assumed to be controllable and is specified by the angle β measured positive counterclockwise from the $+X$ axis. The translational motion is governed by the following equations:

$$\dot{x} = u \tag{1}$$

$$\dot{y} = v \tag{2}$$

$$\dot{u} = \frac{-\mu x}{r^3} + \frac{T}{m_0 - \dot{m}t}\cos\beta \tag{3}$$

$$\dot{v} = \frac{-\mu y}{r^3} + \frac{T}{m_0 - \dot{m}t}\sin\beta \tag{4}$$

where x and y are the satellite's inertial position coordinates, u and v its associated velocity coordinates, μ the universal gravitation constant times the mass of the Earth, r the satellite's distance from the center of the Earth, T the magnitude of the satellite's thrust vector, m the satellite mass, and m_0 its mass at maneuver initiation. The nominal intercept point is arbitrarily chosen as the $+Y$ axis crossing point of the nominal orbit at time t_1 with $v = 0$ and u negative. The evasive maneuver causes the target satellite to depart the nominal orbit at some time t_0 and terminate at t_1. The rendezvous maneuver starts at t_1 and ends at t_2 with the satellite in its nominal position as if no thrusting had occurred between t_0 and t_2. Using continuous thrust, the rendezvous maneuver falls naturally into the context of optimal control theory as a minimum time maneuver, which also minimizes the fuel used. To place the evasive maneuver in the same context, the objective of the maneuver must be quantified. Here we consider two possible objectives of the evasion. Other objectives may be more relevant operationally, but they can be treated in a similar manner. The first objective is to maximize the change in the orbit radius over a fixed interval. The second objective is the same but with the constraint that at t_1 the radius vector must be orthogonal to the velocity vector. These two evasion strategies can be compared in terms of the fuel required to change the orbit radius by a specified amount and then return to the nominal orbit in minimum time. This comparison reveals the extent to which trajectory shaping can be used to trade evasion time for total fuel.

Optimal Control

Following the general development of optimal control theory presented by Bryson and Ho,[1] the objective function or performance index for maximizing the change in orbit radius is

$$J^1 = r(t_1) = \sqrt{x^2(t_1) + y^2(t_1)} \tag{5}$$

Adding the constraint of orthogonality between the final radius vector and velocity vector yields

$$J^2 = r(t_1) + \nu[x(t_1)u(t_1) + y(t_1)v(t_1)] \tag{6}$$

where ν is a constant Lagrange multiplier. For either evasive maneuver, the initiation time t_0 and the state vector $[x\ y\ u\ v]^T$ at t_0 are specified. Using Lagrange multiplier costates $\lambda = [\lambda_x\ \lambda_y\ \lambda_u\ \lambda_v]^T$, the common Hamiltonian for J^1 and J^2 becomes

$$H = \lambda_x u + \lambda_y v + \lambda_u\left[\frac{-\mu x}{r^3} + \frac{T}{m_0 - \dot{m}t}\cos\beta\right] + \lambda_v\left[\frac{-\mu y}{r^3} + \frac{T}{m_0 - \dot{m}t}\sin\beta\right] \tag{7}$$

The optimality condition for the control, $H_\beta = 0$, yields an expression for the thrust angle β:

$$\beta = \tan^{-1}\left[\frac{\lambda_v}{\lambda_u}\right] \tag{8}$$

For J^1 or J^2 to be a maximum, the Legendre-Clebsch condition requires $H_{\beta\beta} \le 0$, which with Eq. (8) implies

$$\cos\beta = \frac{\lambda_u}{\sqrt{\lambda_u^2 + \lambda_v^2}}, \qquad \sin\beta = \frac{\lambda_v}{\sqrt{\lambda_u^2 + \lambda_v^2}} \tag{9}$$

Minimizing J^1 or J^2, however, would require reversing the signs of $\cos\beta$ and $\sin\beta$. The costates λ must satisfy

$$\dot{\lambda}_x = -H_x = \lambda_u\left[\frac{+\mu}{r^3} - \frac{3\mu x^2}{r^5}\right] - \lambda_v\left[\frac{3\mu xy}{r^5}\right] \tag{10}$$

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$$\dot{\lambda}_y = -H_y = \lambda_v \left[\frac{+\mu}{r^3} - \frac{3\mu y^2}{r^5} \right] - \lambda_u \left[\frac{3\mu xy}{r^5} \right] \quad (11)$$

$$\dot{\lambda}_u = -H_u = -\lambda_x \quad (12)$$

$$\dot{\lambda}_v = -H_v = -\lambda_y \quad (13)$$

The boundary conditions on these differential equations for λ are at t_1, and they can be expressed for J^1 and J^2 as

$$\lambda_x(t_1) = x(t_1)/r(t_1) + \nu u(t_1) \quad (14)$$

$$\lambda_y(t_1) = y(t_1)/r(t_1) + \nu v(t_1) \quad (15)$$

$$\lambda_u(t_1) = \nu x(t_1) \quad (16)$$

$$\lambda_v(t_1) = \nu y(t_1) \quad (17)$$

where $\nu = 0$ for J^1, whereas ν is again a constant Lagrange multiplier for J^2.

In the two point boundary value problem (TPBVP) for J^1, the boundary conditions on λ_u and λ_v at t_1 yield indeterminate forms for $\sin\beta$ and $\cos\beta$ at t_1. L'Hospital's Rule[2] can be applied to Eq. (8) at t_1 to yield

$$\beta = \tan^{-1}\left[\frac{-\lambda_y(t_1)}{-\lambda_x(t_1)}\right] \quad (18)$$

At this point, however, $H_{\beta\beta}$ vanishes and the Legendre-Clebsch condition cannot be used to deduce sign explicit expressions for $\cos\beta(t_1)$ and $\sin\beta(t_1)$. Using the boundary condition on λ_x and λ_y at t_1, $\cos\beta(t_1)$ and $\sin\beta(t_1)$ can be expressed as

$$\cos\beta(t_1) = \frac{\pm x(t_1)}{r(t_1)}, \qquad \sin\beta(t_1) = \frac{\pm y(t_1)}{r(t_1)} \quad (19)$$

In Eqs. (19), only the positive or negative sign applies. The positive sign implies that the thrust vector points directly away from the Earth's center at t_1, whereas the negative sign indicates that the thrust vector points directly toward the Earth's center at t_1. To select the proper sign, let $t_0 \rightarrow t_1$ in the limit. Since the objective is to maximize the length of the radius vector at t_1, the direction of thrusting must be directly away from the center of the Earth during an infinitesimally short maneuver. Consequently, the positive sign is appropriate in Eqs. (19) to determine the direction of the control vector at t_1.

In the TPBVP for J^2, ν can be eliminated from Eqs. (14–17) to give three boundary conditions:

$$\lambda_v(t_1)x(t_1) - \lambda_u(t_1)y(t_1) = 0 \quad (20)$$

$$\lambda_x(t_1)x(t_1) - \lambda_u(t_1)u(t_1) - x^2(t_1)/r(t_1) = 0 \quad (21)$$

$$\lambda_y(t_1)y(t_1) - \lambda_v(t_1)v(t_1) - y^2(t_1)/r(t_1) = 0 \quad (22)$$

The fourth required boundary condition is recovered by using the orthogonality condition on the radius vector and velocity vector at t_1,

$$x(t_1)u(t_1) + y(t_1)v(t_1) = 0 \quad (23)$$

Now, for either J^1 or J^2, the state and costate differential equations, the appropriate boundary conditions, and Eqs. (9) form a TPBVP to be solved for the histories of the states and costates. These histories yield the history of β during a maneuver.

For the rendezvous maneuver, the objective function must be augmented with rendezvous constraints using constant Lagrange multipliers $\gamma = [\gamma_x\ \gamma_y\ \gamma_u\ \gamma_v]^T$, and the state equations (1–4) using Lagrange multipliers λ. The rendezvous constraints at t_2 are specified as follows:

$$\psi_x = x(t_2) - x_n = 0 \quad (24)$$

$$\psi_y = y(t_2) - y_n = 0 \quad (25)$$

$$\psi_u = u(t_2) - u_n = 0 \quad (26)$$

$$\psi_v = v(t_2) - v_n = 0 \quad (27)$$

where x_n, y_n, u_n, and v_n are the satellite states in the nominal circular orbit at t_2. Then, representing the state equations (1–4) by the single vector equation $\dot{z} = f(z, \beta, t)$, the objective function for the rendezvous maneuver can be written as

$$J^3 = \gamma^T\psi + \int_{t_1}^{t_2} [1 + \lambda^T(f - \dot{z})]\, dt \quad (28)$$

The Hamiltonian of J^3 may then be defined as

$$H^3 = 1 + \lambda^T f = 1 + H \quad (29)$$

Consequently, the costate equations of the evasion problem also apply to the rendezvous problem. Equation (8), which partially determines the thrust angle β, also applies, but the Legendre-Clebsch condition now requires $H_{\beta\beta} \geq 0$, and, therefore,

$$\cos\beta = -\frac{\lambda_u}{\sqrt{\lambda_u^2 + \lambda_v^2}}, \qquad \sin\beta = -\frac{\lambda_v}{\sqrt{\lambda_u^2 + \lambda_v^2}} \quad (30)$$

There are four boundary conditions at t_1 on the state variables x, y, u, and v. These variables are assigned the same values achieved at t_1 in the evasion maneuver. At t_2, boundary conditions are defined by Eqs. (24–27). The transversality condition[1] determines t_2, yielding

$$H^3(t_2) + [\lambda_x(t_2)y_n(t_2) - \lambda_y(t_2)x_n(t_2) + \lambda_u(t_2)v_n(t_2) - \lambda_v(t_2)u_n(t_2)]v_0/r_0 = 0 \quad (31)$$

The TPBVP for the rendezvous is now summarized as the four state equations (1–4) with β eliminated by Eqs. (30), the four costate equations (10–13), the four boundary conditions on the state variables at t_1, the four boundary conditions at t_2 given by Eqs. (24–27), and the transversality condition given by Eq. (31).

Problem Solution

The optimal control problems outlined in the previous section are nonlinear and must be solved numerically. The method used here solves TPBVPs on a fixed interval by converting differential equations to a larger system of nonlinear algebraic/transcendental equations that are solved simultaneously using Newton's method with deferred corrections. The TPBVPs for the evasion maneuvers are fixed interval problems and are, thus, properly formulated for solution. The TPBVP for the rendezvous maneuver, however, is not a fixed interval problem but can be converted to one in the following way.[3] Let

$$t - t_1 = c\tau \quad (32)$$

with $t_1 \leq t \leq t_2$ and $0 \leq \tau \leq 1$ and c an unknown constant. Then

$$\frac{dt}{d\tau} = c \quad (33)$$

$$\frac{dc}{d\tau} = 0 \quad (34)$$

Now, the TPBVP in t can be converted to a TPBVP in τ. Equation (34) is added to the TPBVP differential equations along with the transversality condition Eq. (31) to give a system of nine differential equations to be solved simultaneously on the fixed interval [0,1].

Results

Evasive maneuvers and nominal orbit rendezvous maneuvers were computed for a 2400-kg geosynchronous spacecraft equipped with 10 30-cm-diam electric propulsion thrusters. The total thrust of this system was 1.3 N from a mass flow rate of 6.9×10^{-5} kg/s. With a nominal intercept point at the $+Y$ axis crossing point of the nominal orbit, evasive maneuvers were computed based on maneuver initiations 2 and 4 h prior to the nominal intercept time. Nominal orbit rendezvous maneuvers were computed from the position, velocity, and mass of the spacecraft at the nominal intercept time. Further information on the following results may be found in Ref. 4.

Solving the TPBVPs derived from J^1 and J^2 yields evasive maneuvers that increase the orbit radius of the target satellite. The changes in orbit radius achieved for the four maneuvers computed are shown in Table 1. Note that the radius increases achieved during the maneuver with the terminal orthogonality constraint are significantly less than those achieved without a terminal constraint.

The control angle profiles for the maximum radius increase maneuvers without the terminal constraint are shown in Fig. 1. Note that the control angle changes very little during the 2- and 4-h maneuvers. The control angle profiles for the maximum radius increase maneuvers with the terminal constraint

Table 1 J^1 and J^2 maximum radius maneuvers

Maneuver length, h	J^1 radius increase, km	J^2 radius increase, km
2	26.5	14.2
4	116.4	64.4

Table 2 Rendezvous times after J^1 and J^2 radius increase maneuvers

Evasive maneuver length, h	J^1 time to rendezvous, h	J^2 time to rendezvous, h
2	8.49	3.6
4	17.79	8.0

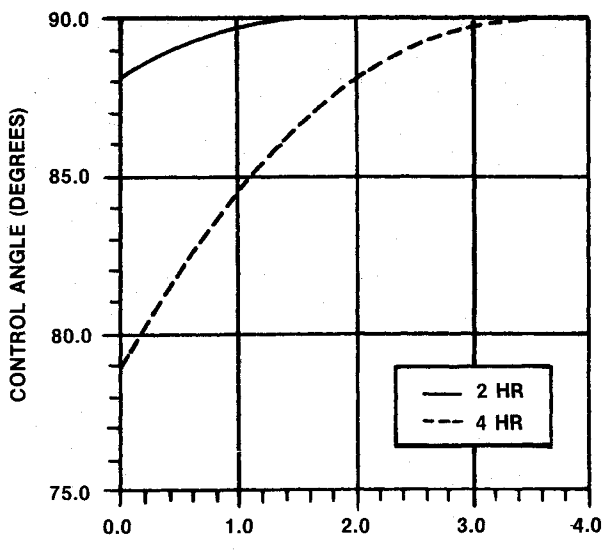

Fig. 1 Control angle vs time for 2- and 4-h unconstrained evasion to maximum radius.

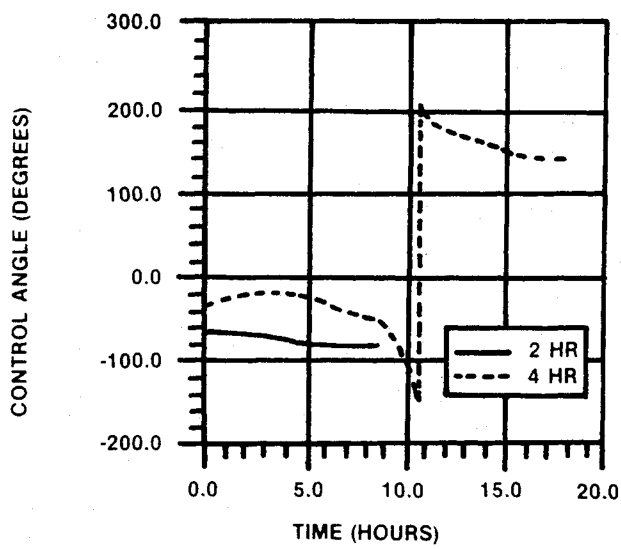

Fig. 3 Control angle vs time for rendezvous after a 2- and 4-h unconstrained evasion.

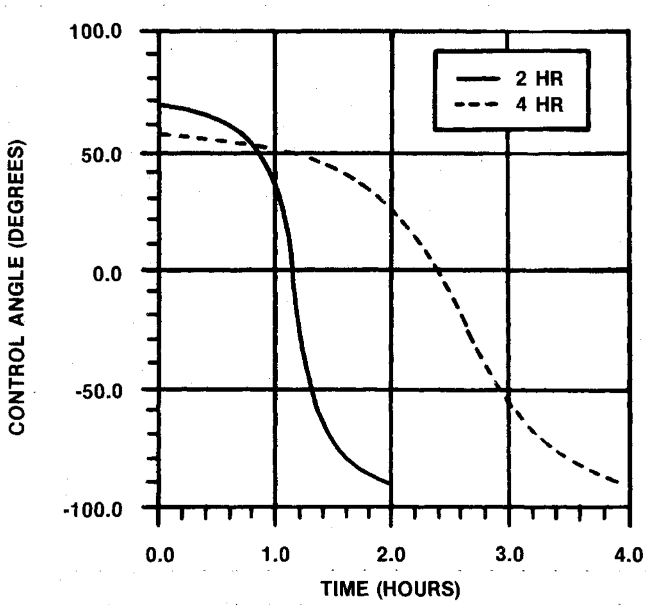

Fig. 2 Control angle vs time for 2- and 4-h constrained evasion to maximum radius.

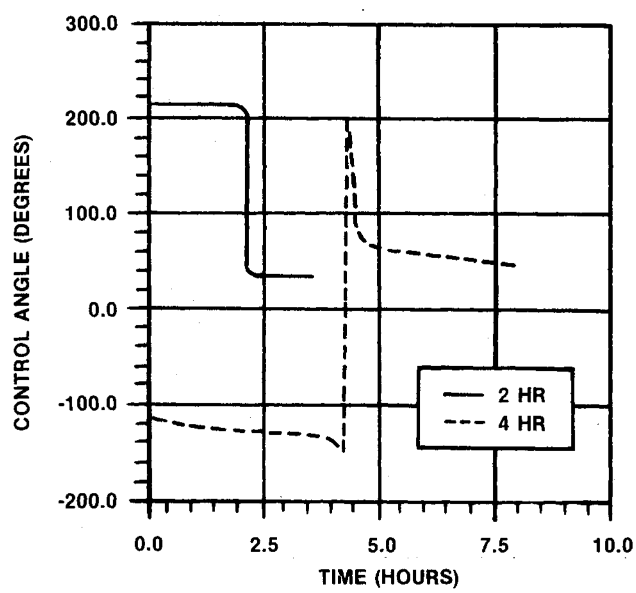

Fig. 4 Control angle vs time for rendezvous after a 2- and 4-h constrained evasion.

are shown in Fig. 2, which can be compared with Fig. 1. Note that each of these maneuvers might be suitably approximated by two constant control angles.

Nominal orbit rendezvous maneuvers from both types of J^1 and J^2 evasive maneuvers were computed by solving the TPBVP derived from J^3. Table 2 summarizes the results achieved when considering nominal orbit rendezvous after maximum radius increase maneuvers both with and without terminal orthogonality constraint. Note that the times to rendezvous are significantly greater than the times to evade given in Table 1, and the times for return from a maneuver without terminal constraint are more than twice as great as those from a maneuver with the terminal orthogonality constraint.

The control histories for rendezvous maneuvers from the evasive maneuvers without terminal constraint are shown in Fig. 3, labeled by associated evasion time. After a 2-h evasion, the control angle varies by approximately 12 deg over the maneuver, with the control angle varying linearly for about 3 h of the maneuver. Figure 3 also shows a highly nonlinear behavior of the control angle with time for the rendezvous from the unconstrained 4-h evasive maneuver, which could be more difficult to implement. The control profiles for rendezvous from evasive maneuvers with terminal constraint are shown in Fig. 4, again labeled by evasion time. Note that these control histories depart significantly from those of Fig. 3, and that two constant control angles approximate each maneuver very well.

With the results achieved, the two evasion strategies can be compared using the two tables to estimate the total time required for evasion and rendezvous. Consider the 2- and 4-h radius increase maneuvers of Table 1. The unconstrained 2-h maneuver increases the orbit radius by 26.5 km. From Table 2, 8.49 h are then required to rendezvous with the nominal orbit for a total maneuver time of 10.49 h. On the other hand, with the terminal othogonality constraint, it appears that the orbit radius can be increased by the same 26.5 km, but in 3 h or less. Then, from Table 2, the subsequent rendezvous maneuver appears to require approximately 6 h for a total maneuver time of approximately 9 h. Consequently, time and fuel can be saved by using any available time to shape the evasive maneuver rather than delaying until a maximum radius increase maneuver is required.

Conclusions

Geosynchronous satellite evasive maneuvers using a constant low thrust magnitude and thrust direction control have been modeled in the context of optimal control theory. With evasive maneuvers restricted to the nominal orbit plane, two maneuver objectives were considered. One maximized the change in the orbit radius. The second did the same but required the final radius and velocity vectors to be orthogonal. Numerical results showed that a considerable penalty was paid in the change of robit radius achievable when the terminal constraint was added. Postevasion rendezvous maneuvers with the nominal orbit were also computed. These results showed that rendezvous times can significantly exceed the times allowed for evasive maneuvers. They also showed that a terminal constraint on the evasive maneuver can be used to save significant time and fuel over the complete maneuver sequence when warning time is greater than the minimum required to evade an intercept by some specified change in the orbit radius.

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Optimal Test Procedures for Evaluating Circular Probable Error

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Introduction

EVALUATING the targeting accuracy of a surface-to-surface missile is a rather complicated decision-making process. Especially given the cost of such systems, the decision should be based on the results of a very limited number of firing tests.

In this Note, we propose several tests through which the claimed circular probable error (CEP) (see Ref. 1) of a given missile can be evaluated. Our approach is based on the following observation. Assuming the claimed CEP of a missile is L, then an ideal test procedure is one for which the probability of passing the test is 1 given the actual CEP is less than or equal to L, and the probability of passing the test is 0 given the actual CEP of missile is greater than L. But an actual test will never have such a sharp operating characteristic (OC) curve (see Ref. 2), and we shall devise test procedures whose objective is to minimize the deviation from this ideal characteristic.

Test Procedures

Let us denote the target point at which all missiles are aimed by T and draw n different circles with centers at T and radii R_i such that

$$R_n \geq \cdots \geq R_2 \geq R_1 \geq 0 \tag{1}$$

Let us place a Cartesian coordinate frame on the target and denote the coordinates of a point with reference to this frame by the ordered pair (x,y). We denote the vector with coordinates (x,y) by r and the impact point of the ith missile fired by r_i. Let S_i denote the set of all points whose distance from T is less than or equal to R_i:

$$S_i = (r \mid |r| \leq R_i) \tag{2}$$

Here $|r|$ denotes the Euclidean norm of the vector r. We denote the complement of S_i by S_i', and this is the set of all points outside a circle with radius R_i.

We define a nonsequential test (NST) with (at most) n firing (NSTn) as follows. Fire n missiles aimed at the same target point T. We shall say the missile qualifies and has the claimed CEP if the following n conditions hold simultaneously:

1) At least one missile from n fired should land at a point in S_1.

2) At least two missiles from n fired should land at points in S_2.

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$n-1$) At least $n-1$ missiles from n fired should land at points in S_{n-1}.

n) All n missiles should land at points in S_n.

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